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On Seifert-manifolds*

by

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This paper is an investigation of certain 3-manifolds (subsequently called Seifert-manifolds or simply manifolds) described by Seifert [7]. The Seifert-manifolds are compact, connected 3-manifolds with the property that they admit a "singular fibering" in the following sense:

the fibers are closed curves,

through each point there is exactly one fiber, and

each fiber has a "fibered neighbourhood".

The latter is a solid cylinder fibered with lines parallel to the axis whose top and bottom are identified such that they have a relative twist of $2\pi\frac{\nu}{\mu}$ before identification. μ and ν are relatively prime integers and we can assume $\mu > 0$ and $0 \leq \nu \leq \frac{\mu}{2}$. The solid torus thus obtained is a "fibered neighbourhood" of its middle fiber (the former cylinderaxis). If $\nu = 0$ choose $\mu = 1$ and call the middle fiber "regular", otherwise call it "exceptional".

Since the manifold M is compact the number of exceptional fibers is finite and $0 = M - \{\text{exceptional fibers}\}$ is a Steenrod fiber bundle which is open and dense in M provided $M \neq 0$. In general it will not be a principal circle bundle. The base space of this fiber bundle is a closed 2-manifold with a point missing for each exceptional fiber.

Seifert [7] classified these manifolds with respect to homeomorphisms which send fibers into fibers (fiber preserving homeomorphisms). Since his methods and results are essential to the understanding of the present work we shall give a brief summary of [7] without proofs. Theorem and page numbers refer to those of [7] unless indicated otherwise. Consider the above mentioned fibered solid cylinder with the appropriate identification of top and bottom. Let m be a meridian, i.e., an oriented simple closed curve on the boundary torus T such that m is not nullhomotopic on T but it is nullhomotopic in the solid torus V. Any two such curves are homologous and there is a fiber preserving homeomorphism of T extendable to V carrying one into the other. Let B denote another simple closed curve on T with the property that B and m form a homology basis on T. Two such curves B_1 and B_2 are not necessarily homologous on T but a fiber preserving homeomorphism of V into V will carry B_1 into B_2.

Let H be the oriented fiber on T and Q an oriented simple closed curve such that H and Q form a homology basis on T. Q is determined modulo H up to orientation.

Clearly $H \sim \nu m + \mu B$ on T

Likewise $m \sim \alpha Q + \beta H$ for some $(\alpha, \beta) = 1$

The pairs (α, β) and (μ, ν) mutually determine each other. (α, β) will essentially characterize the exceptional fiber.

The <u>decomposition space</u> S is a closed 2-manifold connected with (but not imbedded in) the 3-manifold M. To each fiber in M corresponds a point in S. We shall see that a subset of S will in fact be a cross section. In general it will not be extendable to all of S.

The following (Hilfsatz IV p. 164) is needed: given two fibered neighbourhoods of H, V_1 and V_2, there is a fiber preserving homeomorphism of M into M fixed on H and outside some V containing V_1 and V_2, which sends V_1 into V_2. This establishes that M^* obtained from M by deleting the interior of a fibered neighbourhood of a regular fiber is independent of the choice of the fiber and the neighbourhood. Conversely (Hilfsatz VI p. 166): given M^* there is a unique way to sew in a solid torus V such that a curve m given on the boundary torus of M^* (not nullhomologous or homologous to H) becomes nullhomotopic in V. In fact the new Seifert-manifold is completely determined by M^* and the homology class of m on its boundary torus.

A closed path w on S is given the value $+1$ if H preserves its orientation while describing a path in M corresponding to w. If H reverses its orientation w is given the value -1. The values of the generators of the first homology group of S determine the value of any w on S. Clearly deleting solid tori and sewing in solid tori will not change the value of any w since they can be performed away from w. It is therefore meaningful to delete fibered neighbourhoods of all exceptional fibers of M and replace them with regular fibers. Call this new Seifert-manifold M_o, as we shall see it is not uniquely determined. Let M_o be the manifold with boundary (called class space) obtained by deleting an ordinary fibered neighbourhood from M_o. Satz 3 (p. 173) proves that all Seifert-manifolds with given S (and corresponding values on the homology basis of S) can be obtained from M_o by boring out r torusholes and sewing in $r + 1$ arbitrary fibered solid tori, r of which are exceptional and one regular. This makes M_o unique for given M. In fact M_o admits a cross-section as a Steenrod fiber bundle. Taking out r solid tori and sewing in exceptional orbits extends the cross-section to one with singularities - no longer a fiber bundle, of course -

and only the closing up of the final torushole destroys this property.

It remains to find the possible classes of valued decomposition spaces and the possible ways M_o can be closed up to a Seifert-manifold without exceptional fibers. The latter is classified by the number of inequivalent circle bundles over S.

For the orientable Seifert-manifolds the answer is given in Satz 5 (p. 181). There are two classes; if S is orientable all its generators must have value +1 (these manifolds admit a circle action, Raymond [6]) and if S is non-orientable then we can choose a set of homology generators where every one has value -1. Given an orientation the inequivalent circle bundles over S are in a one-to-one correspondance to Z in both classes. The different Seifert-manifolds are obtained by sewing V_o into the last torushole in such a manner that the curve $m_o \sim Q_o + bH$ $b \in Z$ becomes nullhomotopic in V_o . (Q_o and H form a homology basis on the boundary torushole with orientation induced by the manifold.) A complete fiber preserving homeomorphism classification of the orientable Seifert-manifolds with given orientation is:

$$(0,o,g \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$

$$(0,n,k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$

where 0 refers to the orientability of the manifold, o (or n) the orientable (non-orientable) decomposition space of genus g (or k); b is the integer described above, while the pairs (α_j, β_j) prescribe the filling of the j-th torushole such that

$m_j \sim \alpha_j Q_j + \beta_j H$ becomes nullhomotopic in V_j (where Q_j and H form a homology basis on the j-th torushole). Due to the fact that H is a simple closed curve $(\alpha_j, \beta_j) = 1$, and since Q_j is only determined modulo H, β_j can be normalized such that $0 < \beta_j < \alpha_j$.

Reversal of the orientation of M gives:

$$(0,o,g \mid -r-b; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r)$$

$$(o,n,k \mid -r-b; \alpha_1, \alpha_1 - \beta_1; \ldots; \alpha_r, \alpha_r - \beta_r)$$

The non-orientable Seifert-manifolds admit four classes. If S is orientable a homology basis can be chosen such that each generator has value -1. If S is non-orientable, and M_o is a product bundle (i.e., +1 for all generators) then the class is called N,nI. Again it admits a circle action, Raymond [6] . In case there exist curves on S with value -1, it is shown that the number of generators with value $+1$ can be decreased by suitable choice of new generators until there are at most two left. This leaves the possibilities of N,nII with exactly one generator with value $+1$ all others $-1 (k \geq 2)$ and N,nIII with exactly two generators with value +1 all others $-1 (k \geq 3)$. (A summary of the possible classes is given in Satz 7 p. 188). When M is non-orientable the ways of closing up M_o is in one-to-one correspondance with Z_2 if there are no exceptional fibers of multiplicity two (i.e., no $\alpha_j = 2$), and if some $\alpha_j = 2$ there is only one way to do it.

The Q_j are only determined modulo H up to orientation, but since M is nonorientable there is no preferred orientation on the torusholes,, hence we can normalize β_j such that

$0 < \beta_j \leq \alpha_j/2$ ($\beta_j = \alpha_j/2$ if and only if $\alpha_j = 2$, since $(\alpha_j, \beta_j) = 1$ is still required.) The non-orientable Seifert-manifolds are therefore by Satz 8 (p. 193):

$$(N,o,g \mid b, \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$
$$(N,nI,k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r)$$
$$(N,nII,k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \qquad k \geq 2$$
$$(N,nIII,k \mid b; \alpha_1, \beta_1; \ldots; \alpha_r, \beta_r) \qquad k \geq 3$$

where $b = 0$ or 1 if no $\alpha_j = 2$; otherwise immaterial.

This is the main result of [7]. The fundamental groups are computed on p. 200. The Seifert-manifolds with finite fundamental groups are listed on p. 203. These are treated in detail in [8] where they are shown to coincide with the orbit spaces of fixedpoint free orthogonal group actions of finite groups on S^3.

Results relevant to the topological classification will be quoted in the text. It is interesting to note (Satz 11, p. 206) that the Poincaré conjecture can be proved for Seifert-manifolds, i.e., every simply connected Seifert-manifold is S^3.

The Seifert-manifolds are of interest because they are (except $P^3 \# P^3 \cong (0,n,1 \mid 0)$) prime manifolds, i.e., not the connected sum of two manifolds each different from S^3 ([7] Satz 22, p. 229). Due to Kneser [2] and Milnor [4] we know that every 3-manifold can be written uniquely as the connected sum of prime manifolds with $S^2 \times S^1$ or the corresponding non-orientable handle N. This means that the Seifert-manifolds are "building blocks" in this decomposition.

Moreover all Seifert-manifolds with infinite fundamental group are $K(\pi,1)$-s. Since $\pi_2(M) = 0$ (except $S^2 \times S^1$ and N) this combined with infinite $\pi_1(M)$ implies that the universal covering space of M (say $\tilde{M}$) is contractible. $\pi_2(\tilde{M}) = 0$ and $\pi_1(\tilde{M}) = 0$ and the fact that $\tilde{M}$ is not compact show $H_3(\tilde{M}) = 0$ and $H_n(\tilde{M}) = 0 \quad n > 3$. Hence all higher homotopy groups vanish and $\pi_1(M)$ determines the homotopy type of M.

The only work available to the author concerning the topological classification of these manifolds is a paper by Brody [1] , where the class 0,o,g is studied for $g > 0$, $r = 0$, 1 or 2. In case $r = 2$ certain requirements on α_1, α_2 are necessary for the topological classification. The use of Alexander matrices and elementary ideals in this connection is due to Brody.

The main results* of this investigation are

<u>Theorem 1</u>: Two Seifert-manifolds with $g > 0$ are homeomorphic only if they are of the same class and have the same genus, except:

$(N,o,1|0)$ is homeomorphic to $(N,nI,2|0)$ and

the question whether

$(N,o,1|1)$ is homeomorphic to $(N,nI,2|1)$ and

$(N,nII,2|1)$ to $(N,nI,1|-;\ 2,1,;2,1)$ is undecided.

<u>Theorem 2</u>: Two Seifert-manifolds of the same class and genus with $g > 0$ and $r > 1$ are homeomorphic if and only if they are fiber-preserving homeomorphic.

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*) These results have been obtained independently by E. Vogt and H. Zieschang. Joint publication is under preparation.

The above results settle the topological classification for the majority of Seifert-manifolds.

We shall establish both theorems through an analysis of the fundamental groups. Here

$$\pi_1(M) = (C_i, Q_j, H \mid [C_1, C_2] \ \ldots \ [C_{2g-1}, C_{2g}] \ Q_r^{-1} \ldots Q_1^{-1} H^b; \quad Q_j^{\alpha_j} H^{\beta_j}; \ C_i H C_i^{-1} H^{\epsilon_i}; \ [Q_j, H])$$

with $\epsilon_i = -1$ for all i if $M = (O, o, g \mid \ldots)$

$\epsilon_i = 1$ for all i if $M = (N, o, g \mid \ldots)$

$i = 1, 2, \ldots, 2g \qquad j = 1, 2, \ldots, r.$

let here $F = [C_1, C_2] \ \ldots \ [C_{2g-1}, C_{2g}] \ Q_r^{-1} \ \ldots \ Q_1^{-1}$.

$$\pi_1(M) = (A_i, Q_j, H \mid A_1^2 A_2^2 \ldots A_k^2 \ Q_r^{-1} \ldots Q_1^{-1} \ H^b; \ Q_j^{\alpha_j} H^{\beta_j}; A_i H A_i^{-1} \ H^{\epsilon_i}; \ [Q_j, H])$$

with $\epsilon_i = 1$ for all i if $M = (O, n, k \mid \ldots)$

$\epsilon_i = -1$ for all i if $M = (N, nI, k \mid \ldots)$

$\epsilon_i = 1$ for $i = 1, 2, \ldots, k-1$; $\epsilon_k = -1$ if $M = (N, nII, k \mid \ldots)$

$\epsilon_i = 1$ for $i = 1, 2, \ldots, k-2$; $\epsilon_{k-1} = \epsilon_k = -1$ if $M = (N, nIII, k \mid \ldots$

$i = 1, 2, \ldots, k; \qquad j = 1, 2, \ldots, r.$

let here $P = A_1^2 \ \ldots \ A_k^2 \ Q_r^{-1} \ \ldots \ Q_1^{-1}$

The proof of <u>Theorem 1</u> proceeds through a series of technical Lemmas. The basic tools are the first homology groups and the elementary ideals via the Alexander matrices of the fundamental groups. Clearly orientable and non-orientable Seifert-manifolds are distinguished by their third homology groups thus they can be treated separately. As an illustration we shall state the necessary lemmas in the orientable case:

Lemma 1: If $g_1 \neq g_2$ then the Seifert-manifolds $M_1 = (0,o,g_1 | \ldots)$ and $M_2 = (0,o,g_2 | \ldots)$ are not homeomorphic.

Proof: $H_1(M_1) \neq H_1(M_2)$

Lemma 2: If $k_1 \neq K_2$ then the Seifert-manifolds $M_1 = (0,n,k_1 | \ldots)$ and $M_2 = (0,n,k_2 | \ldots)$ are not homeomorphic.

Proof: $H_1(M_1) \neq H_1(M_2)$

Lemma 3: The Seifert-manifolds $M_1 = (0,o,\cdot | \ldots)$ and $M_2 = (0,n,\cdot | \ldots)$ are not homeomorphic.

Proof: $H_1(M_1) = H_1(M_2)$ implies $2g = k-1$ but $E^1_{2g-1} \neq 0$ while $E^2_{k-2} = 0$.

Here E^1_{2g-1} is the $(2g-1)$-st elementary ideal of $\pi_1(M_1)$ and E^2_{k-2} is the $(k-2)$-nd elementary ideal of $\pi_1(M_2)$. The computation is found in [5].

We omit the non-orientable case which is essentially the same except for a few delicate points.

The proof of Theorem 2 will proceed in several steps. First we establish that H is a distinguished generator of $\pi_1(M)$, next we shall study $\pi_1(M)/(H)$ to conclude that the set $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ is necessarily the same for two homeomorphic Seifert-manifolds and finally use some results from the theory of non-Euclidean crystallographic groups to show that modulo orientation the β_j and b have to agree, too.

In the classes $(O,o,\cdot \mid \ldots)$ and $(N,nI,\cdot \mid \ldots)$ the subgroup generated by H is in the center and it is easy to prove that the group $\pi_1(M)/(H)$ - which is an amalgamated free product - has no center. Thus H is distinguished.

In the other four classes H^2 generates the center and H generates the only maximal cyclic normal subgroup. The latter requires some work.

The groups $\pi_1(M)/(H)$ are fairly well understood. For $(O,o,g \mid \ldots)$ and $(N,o,g \mid \ldots)$

$$\pi_1(M)/(H) = (C_i, Q_j \mid F; Q_j^{\alpha_j})$$

which is a Fuchsion group, and for the other classes

$$\pi_1(M)/(H) = (A_i, Q_j \mid P; Q_j^{\alpha_j})$$

The $(\alpha_1, \ldots, \alpha_r)$ are the orders of the fixed points of the non-Euclidean rotations and thus they are invariants of the quotiunt groups and thus of $\pi_1(M)$. To conclude the same about the β_j and b we invoke a result of H. Zieschang [9] :

<u>Theorem</u>: Let $G = (C_i, Q_j \mid [C_1, C_2] \ \ldots \ [C_{2g-1}, C_{2g}] \ Q_r^{-1} \ldots Q_1^{-1}; Q_j^{\alpha_j})$ and let $\hat{G} = (\hat{C}_i, \hat{Q}_j \mid \quad)$ be the free group generated by the C_i and the Q_j. Every automorphism η of G is induced by an automorphism $\hat{\eta}$ of $\hat{G}$ with the following properties:

$$\hat{\eta}(\hat{Q}_j) = \hat{M} \hat{Q}_{\nu_j}^{\pm 1} \hat{M}^{-1} \quad \text{where} \quad \alpha_{\nu_j} = \alpha_j \text{ and } \hat{M} \text{ is some word.}$$

$$\hat{\eta}(\hat{F}) = \hat{N} \hat{F}^{\pm 1} \hat{N}^{-1} \quad \text{where } \hat{N} \text{ is some word}$$

$$\text{and } \hat{F} = [\hat{C}_1, \hat{C}_2] \ldots [\hat{C}_{2g-1}, \hat{C}_{2g}] \ \hat{Q}_r^{-1} \ldots \hat{Q}_1^{-1}$$

Due to the generalized Nielsen theorem of Macbeath [3] this theorem extends to G with non-orientable fundamental region.

Assume now that two Seifert-manifolds M and $\bar{M}$ are homeomorphic. The given map induces an isomorphism of their fundamental groups

$$\varphi : \pi_1(M) \longrightarrow \pi_1(\bar{M}) \qquad \text{iso}$$

where $\varphi(H) = \bar{H}^{\pm 1}$ since H is a distinguished generator. Thus φ in turn induces an isomorphism on the quotient group level

$$\eta : \pi_1(M)/(H) \longrightarrow \pi_1(\bar{M})/(\bar{H})$$

which we can consider an automorphism as well.

We have the commutative diagram

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{d} & \pi_1(M)/(H) \\ \varphi \downarrow & & \downarrow \eta \\ \pi_1(\bar{M}) & \xrightarrow{\bar{d}} & \pi_1(\bar{M})/(\bar{H}) \end{array}$$

where d is the quotient map.

By Zieschang's theorem η is induced by some $\hat{\eta}$ on the free group level with given properties.

Let now $\hat{K} = (\hat{C}_i, \hat{Q}_j, \hat{H} \mid \hat{C}_i \hat{H} \hat{C}_i^{-1} \hat{H}^{\epsilon_i} ; \; [\hat{Q}_j, \hat{H}])$

with the ϵ_i corresponding to the ϵ_i in $\pi_1(M)$,

$f: \hat{K} \longrightarrow \pi_1(M)$ by dropping hats and

$f_1: \hat{G} \longrightarrow \pi_1(M)/(H)$ its restriction and

$D: \hat{K} \longrightarrow \hat{G}$ the quotient map by $(\hat{H})$

Then we have the commutative diagram:

$$\begin{tikzcd}
\hat{K} \arrow[rrr, "D"] \arrow[dr, "f"] \arrow[ddd, dashed, "\hat{\varphi}"'] & & & \hat{G} \arrow[dl, "f_1"'] \arrow[ddd, "\hat{\eta}"] \\
& \pi_1(M) \arrow[r, "d"] \arrow[d, "\varphi"] & \pi_1(M)/(H) \arrow[d, "\eta"] & \\
& \pi_1(\bar{M}) \arrow[r, "\bar{d}"] & \pi_1(\bar{M})/(\bar{H}) & \\
\hat{\bar{K}} \arrow[rrr, "\bar{D}"] \arrow[ur, "\bar{f}"] & & & \hat{\bar{G}} \arrow[ul, "\bar{f}_1"']
\end{tikzcd}$$

and we are looking for a map $\hat{\varphi}: \hat{K} \longrightarrow \hat{\bar{K}}$ to complete the diagram.

Define $\hat{\varphi}_1 = p\,\hat{\eta}\,D$ where

$p: \hat{\bar{G}} \longrightarrow \hat{\bar{K}}$ by mapping each generator of $\hat{\bar{G}}$ into the corresponding generator of $\hat{\bar{K}}$. This will make the diagram commutative modulo $\ker \bar{d} = (\bar{H})$.

Notice that every word in $\hat{K}$ is of the form $\hat{W} = \hat{H}^m \hat{W}_1$ where $\hat{W}_1$ no longer contains $\hat{H}$. Suppose we have

$\varphi f(\hat{W}) = \bar{H}^{w} \ \bar{f} \hat{\varphi}_1(\hat{W})$, then we define

$\hat{\varphi}(\hat{W}) = \bar{H}^{w} \ \hat{\varphi}_1(\hat{W})$

The map $\hat{\varphi}$ thus defined actually completes the diagram and forces φ to have the properties:

$$\varphi(Q_j) = \bar{H}^{q_j} \bar{M} \bar{Q}_{\nu_j}^{\pm 1} \bar{M}^{-1} \qquad \alpha_{\nu_j} = \alpha_j$$

$$\varphi(FH^{b}) = \bar{H}^{s} \bar{N} \bar{F} \bar{H}^{b} \bar{N}^{-1}$$

or $$\varphi(PH^{b}) = \bar{H}^{m} \bar{N} \bar{P} \bar{H}^{b} \bar{N}^{-1}$$

where s and m are even modulo $\sum_{j=1}^{r} q_j$ [illegible]

and depend on the sign of $\bar{Q}_{\nu_j}$, the ϵ_i and the sign of $\varphi(H) = \bar{H}^{\pm 1}$. It is only a matter of computation to show that for M orientable $\beta_j = \bar{\beta}_j$ or $\bar{\alpha}_j - \bar{\beta}_j$ and for M nonorientable $\beta_j = \bar{\beta}_j$. Likewise $b = \bar{b}$ or $b = -r-\bar{b}$ in the orientable case and $b = \bar{b}$ otherwise.

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